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Real-time Infinite Horizon Linear-Quadratic Tracking Controller for Vibration Quenching in Flexible Beams

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Abstract

It is well known that the infinite horizon linear-quadratic tracking problem does not have a solution in the strict sense because in general the performance index is infinite. Additional computational difficulties arise for applications where real-time computations are needed. However, for applications where the reference signal is generated by an asymptotically stable system, then the problem has an optimal solution. In the present paper a methodology that can be used to design real-time infinite horizon linear-quadratic tracking controllers for vibration quenching in flexible beams is presented. A numerical example is included to illustrate the effectiveness of the method.

1. Introduction

The development of the theory of optimal linear-quadratic regulators, tracking, and servo controllers has become standard textbook material [1, 2, 3]. The steady-state or infinite horizon linear-quadratic *regulator* for linear, time-invariant systems is particularly appealing and always used in practice since its solution, a constant gain, stabilizing, full-state feedback control, is obtained by solving the algebraic Riccati equation (ARE). The infinite horizon or steady-state linear-quadratic *tracking* controller (LQT), on the other hand, has received much less attention mainly because for most reference signals, the cost of tracking becomes unbounded and there are additional computational difficulties that are not present in the regulator.

It is well known that the infinite horizon LQT problem does not have a solution in the strict sense because in general the performance index is infinite. However, for applications where the reference signal is generated by an asymptotically stable system, the problem is well posed and has an optimal solution. An additional computational difficulty arises for applications where real-time computations are needed. This is due to the term that involves an auxiliary steady-state function $v_{ss}(t)$, which is determined by integrating a differential equation with boundary condition, backwards in time. The focus in the present work is to show an alternate technique for designing and implementing infinite horizon LQT controllers for the type of applications where real-time computations are required, and where model-following reference signals can be used. In particular, it is shown that this technique can be effectively used for vibration suppression in flexible beams. We emphasize the work reported at the present time is only theoretical, however computer simulations with numerical values are included to clarify the theory. The remainder of the article is organized as follows: Section 2 defines the LQT problem, in Section 3 the infinite horizon LQT case is considered, in Section 4 the expressions required for a real-time controller under the LQT framework are derived, in Section 5 the proposed technique is applied to the suppression of vibrations in a flexible beam, and in Section 6 the conclusions are provided.

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2. The Finite Horizon LQT Controller

Consider the linear system

$$\begin{aligned} \dot{x}(t) &= Ax(t)+Bu(t), \qquad x(t_0)=x_0 \quad \text{given} \\ y(t) &= Cx(t) \end{aligned} \tag{1}$$

being $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, and $y \in \mathbb{R}^r$, the state, control and output vectors, and the quadratic performance index

$$\begin{aligned} J(u) = & \frac{1}{2}[(x(T)-x_r(T))'Q_f(x(T)-x_r(T)] \\ & + \frac{1}{2}\int_{t_0}^{T}[(x(t)-x_r(t))'Q(x(t)-x_r(t)) \\ & + u'(t)Ru(t)]dt \end{aligned} \tag{2}$$

The initial time is t_0, and the final time is T. The symmetric control and state weighting matrices, R and Q, are chosen by the designer to ensure appropriate penalties for control and tracking error costs. $(\cdot)'$ denotes the transpose of the indicated vector or matrix. The pair $\{A,B\}$ is assumed to be controllable, and $\{A,C\}$ is observable. The state trajectory, $x_r(t)$, is related to the desired output trajectory, $y_r(t)$, by

$$x_r(t) = \mathscr{L} y_r(t); \quad \mathscr{L} = C'(CC')^{-1} \tag{3}$$

Q is nonnegative definite symmetric matrix and satisfies:

$$Q = \overline{C}Q_1\overline{C'} + C'Q_2C; \quad \overline{C'} = I_n - \mathscr{L}C \tag{4}$$

Selection of the matrices Q_1 and Q_2 has to be made. It may be necessary to try a range of values to select the particular one that is most appropriate for a given application.

In the *finite* horizon tracking problem one synthesizes the control law which minimizes the finite-time performance index (2) subject to the constraint (1), where $x_r(t)$ is the reference state trajectory, satisfying (3). $Q \geq 0$, is real symmetric and satisfies (3). $Q_f \geq 0$, and $R > 0$ are real symmetric matrices.

The optimal control law consists of the sum of two components given by

$$u(t) = -K(t)x(t) + R^{-1}B'v(t) \tag{5}$$

where the first term is a full-state feedback with Kalman gain $K(t) = R^{-1}B'P(t)$, and $P(t)$ satisfies the matrix differential Riccati equation

$$\begin{aligned} -\dot{P}(t) &= A'P(t) + P(t)A \\ &\quad - P(t)BR^{-1}B'P(t) + Q \\ P(T) &= Q_f \end{aligned} \tag{6}$$

and the second term uses the *auxiliary* function $v(t)$ which is found from the solution of the differential equation

$$\begin{aligned} -\dot{v}(t) &= [A - BK(t)]'v(t) - Qx_r(t) \\ v(T) &= Q_f x_r(T) \end{aligned} \tag{7}$$

The closed-loop system under the influence of this tracking control law is

$$\begin{aligned} \dot{x}(t) &= [A - BK(t)]x(t) \\ &\quad + BR^{-1}B'v(t) \end{aligned} \tag{8}$$

and the optimal cost on $[t, T]$ for any t using this control is

$$J^*(t) = x'(t)P(t)x(t) + 2x'(t)v(t) + w(t) \tag{9}$$

where the new function $w(t)$ satisfies

$$\begin{aligned} -\dot{w}(t) &= x_r'(t)Qx_r(t) \\ &\quad - \frac{1}{2}v'(t)BR^{-1}B'v(t) \end{aligned} \tag{10}$$

with boundary condition: $w(T) = \frac{1}{2}x_r'(T)Q_f x_r(T)$

2.1. Implementation of the Tracking Controller

The implementation of the tracker (5) in real-time involves a standard optimal feedback regulator and a feed-forward controller. The feedback regulator term requires the backward-in-time solution of the Riccati equation (6). Since the solution of $P(t)$ is independent of the reference state trajectory, the Riccati equation can be solved off-line first and the feedback gain $K(t)$ can be stored. For particular applications where the reference signal $x_r(t)$ is known a priori over the interval $[t, T]$, then the auxiliary function $v(t)$, can be computed off-line by integrating, backwards in time, the differential equation (7), then the initial value of the auxiliary function, $v(0)$ is known, and during the actual control run, $v(0)$ can be used to solve a forward differential equation instead.

3. The Infinite Horizon LQT Controller

In this paper, we restrict our attention to the infinite horizon version of the LQT problem. We consider linear, time-invariant plants and a quadratic performance index with time invariant parameters, and infinite terminal time. In addition, we let $t_0 = 0$, and $Q_f = 0$. Assuming that $\{A, B\}$ is controllable, $\{A, C\}$ is observable, then the Riccati equation solution reaches a steady-state solution $P_{ss} = P(\infty)$, the Kalman gain reaches also a steady-state value $K_{ss} = K(\infty)$, and the closed-loop system matrix

$$A_c = (A - BK_{ss}) \tag{11}$$

is asymptotically stable. Under these conditions, the solution to the infinite horizon LQT problem is given by

$$u(t) = -K_{ss}x(t) + R^{-1}B'v_{ss}(t) \tag{12}$$

where $K_{ss} = R^{-1}B'P_{ss}$ and P_{ss} is the solution to the *Algebraic Riccati Equation* (ARE)

$$0 = A'P_{ss} + P_{ss}A - P_{ss}BR^{-1}B'P_{ss} + Q$$

and we define the limiting function

$$v_{ss}(t) \stackrel{\triangle}{=} v(t)|_{T\to\infty}$$

In this case, the state feedback part of the controller (12), becomes time invariant. However, the component where the term $v_{ss}(t)$ appears, is in general time varying, and a theoretical difficulty arises in the solution of the infinite horizon LQT problem. In essence, as $T \to \infty$, the cost functional (9) is in general unbounded and the meaning of minimality is lost, that is, strictly speaking, the optimal control does not exist. However there are certain cases where the infinite horizon tracking problem is well-posed and has an optimal solution. For example, for the class of reference signals generated by asymptotically stable systems. A practical example is the stabilization of vibratory modes in flexible structures. Here, it is natural to consider reference signals that are asymptotically stable such as damped sinusoids. On the other hand, the proposed design framework may still be found useful to obtain an implementable control law even if it is not optimal in the strict sense. The infinite horizon tracker can be designed for a finite control interval $[0, T]$ by using the steady-state gain K_{ss} and the auxiliary function $v_{ss}(t)$. In general, this sub-optimal tracker is satisfactory if $[T - t_0]$ is large.

4. Real-time Controllers Under the Infinite Horizon LQT Framework

In this section the required expressions to compute two initial conditions, $v_{ss}(0)$ and $w_{ss}(0)$, are derived. In [4] the authors present a different approach to evaluate these initial conditions. It is important to emphasize that these expressions are needed for the design of real-time controllers under the infinite horizon LQT framework.

4.1. Computation of $v_{ss}(t)$

Consider the infinite horizon LQT problem:

System

$$\begin{aligned} \dot{x}(t) &= Ax(t)+Bu(t), \qquad x(0)=x_0 \quad \text{given} \\ y(t) &= Cx(t) \end{aligned} \tag{13}$$

Performance Index

$$\begin{aligned} J(u) &= \frac{1}{2}\int_0^{\infty}[(x(t)-x_r(t))'Q(x(t)-x_r(t)) \\ &\quad + u(t)'Ru(t)dt] \end{aligned} \tag{14}$$

The relationship between the state trajectory, $x_r(t)$, and the desired output trajectory, $y_r(t)$, is given in eq. (3), and the nonnegative definite symmetric matrix Q satisfies eq. (4).

Optimal Controller

$$u(t) = -K_{ss}x(t) + R^{-1}B'\,v_{ss}(t) \tag{15}$$

where $K_{ss} = R^{-1}B'P_{ss}$, and P_{ss} satisfies the ARE.

Auxiliary Differential Equation

$$-\dot{v}_{ss}(t) = [A-BK_{ss}]'\,v_{ss}(t) - Qx_r(t);\, v_{ss}(T)=0 \tag{16}$$

where the dimension of the vectors and matrices are:
$x \in \mathbb{R}^n, x_r \in \mathbb{R}^n, u \in \mathbb{R}^m, y \in \mathbb{R}^r, y_r \in \mathbb{R}^r, A: n\times n,$
$B: n\times m, C: r\times n, R: m\times m, \mathscr{L}: n\times r, Q_1: n\times n,$
$Q_2: r\times r, Q: n\times n, \overline{C}: n\times n, I_n: n\times n$ (unit matrix)

Assume that we are given a r-vector reference signal y_r, which is the output of the known linear reference model

$$\begin{aligned} \dot{z}(t) &= A_r z(t), \qquad z(0)=z_0 \\ y_r(t) &= C_r z(t) \end{aligned} \tag{17}$$

where $z\in\mathbb{R}^{n_r}, y_r\in\mathbb{R}^r, A_r: n_r\times n_r, C_r: r\times n_r$
Define the new variable X as:

$$X = [v_{ss} \qquad z]' \tag{18}$$

using the following equations and eqs. (3) and (17)

$$\begin{aligned} \dot{v}_{ss}(t) &= -A_c'\,v_{ss}(t) + Qx_r(t); \\ \dot{z}(t) &= A_r z(t) \end{aligned} \tag{19}$$

where A_c is the closed-loop gain (see eq.(11)), we assemble the augmented system

$$\dot{X} = \begin{bmatrix} -A_c' & Q_r \\ 0 & A_r \end{bmatrix} X = A_{cr}X \tag{20}$$

being $Q_r = Q\mathscr{L}C_r$.

A transformation matrix, $\mathscr{T}$, can be found such that a block diagonal matrix can be obtained. This is, $X = \mathscr{T}\overline{X}$, and the block diagonal matrix is

$$\dot{\overline{X}} = \begin{bmatrix} -\Lambda_c & 0 \\ 0 & A_r \end{bmatrix} \overline{X} \tag{21}$$

being Λ_c diagonal. Because $\dot{X} = \mathscr{T}\dot{\overline{X}}$ and eq.(20), we have $\mathscr{T}\dot{\overline{X}} = A_{cr}\mathscr{T}\overline{X}$, and therefore

$$A_{cr}\mathscr{T} = \mathscr{T}\begin{bmatrix} -\Lambda_c & 0 \\ 0 & A_r \end{bmatrix} \tag{22}$$

Furthermore, the matrix $\mathscr{T}$ can be partitioned as

$$\mathscr{T} = \begin{bmatrix} \mathscr{T}_1 & \mathscr{T}_2 \\ \mathscr{T}_3 & \mathscr{T}_4 \end{bmatrix} \tag{23}$$

where the partitioned matrices $\mathscr{T}_1, \mathscr{T}_2, \mathscr{T}_3$, and $\mathscr{T}_4$ are $(n\times n), (n\times n_r), (n_r\times n)$, and $(n_r\times n_r)$ respectively.

Then, we have

$$\begin{bmatrix} -A_c' & Q_r \\ 0 & A_r \end{bmatrix}\begin{bmatrix} \mathscr{T}_1 & \mathscr{T}_2 \\ \mathscr{T}_3 & \mathscr{T}_4 \end{bmatrix} = \begin{bmatrix} \mathscr{T}_1 & \mathscr{T}_2 \\ \mathscr{T}_3 & \mathscr{T}_4 \end{bmatrix}\begin{bmatrix} -\Lambda_c & 0 \\ 0 & A_r \end{bmatrix} \tag{24}$$

This yields the following matrix equations

$$\begin{aligned} -A_c'\mathscr{T}_1 + Q_r\mathscr{T}_3 &= -\mathscr{T}_1\Lambda_c \\ -A_c'\mathscr{T}_2 + Q_r\mathscr{T}_4 &= \mathscr{T}_2 A_r \\ A_r\mathscr{T}_3 &= -\mathscr{T}_3\Lambda_c \\ A_r\mathscr{T}_4 &= \mathscr{T}_4 A_r \end{aligned} \tag{25}$$

From here we have that $\mathscr{T}_3 = [0]$, $\mathscr{T}_4 = I_{n_r}$ (unit matrix), and

$$\mathscr{T} = \begin{bmatrix} \mathscr{T}_1 & \mathscr{T}_2 \\ 0 & I_{n_r} \end{bmatrix};\, \mathscr{T}^{-1} = \begin{bmatrix} \mathscr{T}_1^{-1} & -\mathscr{T}_1^{-1}\mathscr{T}_2 \\ 0 & I_{n_r} \end{bmatrix} \tag{26}$$

being $\mathscr{T}_1$ the matrix that diagonalizes A_c'.
Note that Λ_c has stable eigenvalues (eigenvalues of the closed-loop system), and $-\Lambda_c$ has *unstable eigenvalues*, $\sigma(-\Lambda_c)\in\mathbb{C}^+$.
The matrix $\mathscr{T}_2$ can be found by solving the matrix equation

$$A_c'\mathscr{T}_2 + \mathscr{T}_2 A_r = -Q_r \tag{27}$$

A solution exists for this equation if $\sigma(A_r)\cap\sigma(A_c)=0$, and we can always choose a reference model to satisfy this equation.

After applying the transformation matrix $\mathscr{T}$ we can solve for $\overline{X}$, obtaining two independent matrix differential equations: $\dot{\overline{X}}_1 = -\Lambda_c\overline{X}_1$ and $\dot{\overline{X}}_2 = A_r\overline{X}_2$, with solution $\overline{X}_1 = e^{-\Lambda_c t}\,\overline{X}_1(0)$ and $\overline{X}_2 = e^{A_r t}\,\overline{X}_2(0)$ respectively.

Using the equation $X = \mathscr{T}\overline{X}$ we obtain the system

$$\begin{bmatrix} v_{ss} \\ z \end{bmatrix} = \begin{bmatrix} \mathscr{T}_1 & \mathscr{T}_2 \\ 0 & I_{n_r} \end{bmatrix}\begin{bmatrix} \overline{X}_1 \\ \overline{X}_2 \end{bmatrix}$$

we have $v_{ss} = \mathscr{T}_1\overline{X}_1 + \mathscr{T}_2\overline{X}_2$, and we obtain

$$v_{ss} = \mathscr{T}_1 e^{-\Lambda_c t}\,\overline{X}_1(0) + \mathscr{T}_2 e^{A_r t}\,\overline{X}_2(0) \tag{28}$$

We choose our reference system such that $\sigma(A_r)\in\mathbb{C}^-$. Because $\sigma(-\Lambda_c)\in\mathbb{C}^+$, and we want v_{ss} to be a stable signal, then we need to cancel the unstable term. This can be done for example, forcing $\overline{X}_1(0)=0$.

We have that $\overline{X} = \mathscr{T}^{-1}X$, then we obtain

$$\overline{X}_1(0) = \mathcal{T}_1^{-1} v_{ss}(0) - \mathcal{T}_1^{-1}\mathcal{T}_2 z(0) \quad (29)$$
$$\overline{X}_2(0) = z(0) \quad (30)$$

Because $\overline{X}_1(0) = 0$, from eq. (29) we have that
$$v_{ss}(0) = \mathcal{T}_2 z(0) \quad (31)$$

Note that using this equation, the initial condition $v_{ss}(0)$, of the auxiliary equation (16) can be computed, avoiding in this way to solve the differential equation backwards in time. The importance of eq. (31) is that it allows to compute real-time infinite horizon LQT contollers. To use expression (31) we need to know the initial condition $z(0)$ of the reference model, and solve the matrix equation (27).

Using the argument that $v_{ss}(t)$ has to be a stable signal in order to have a bounded control $u(t)$, and hence a finite cost J, we have that
$$v_{ss}(t) = \mathcal{T}_2 z(t) \quad (32)$$

4.2. Minimum Cost and Computation of w_{ss}

For the infinite horizon LQT problem the minimum cost is given by
$$J^* = x_0' P_{ss} x_0 + 2x_0' v_{ss}(0) + w_{ss}(0) \quad (33)$$
where the function w_{ss} satisfies the differential equation
$$-\dot{w}_{ss} = v_{ss}' BR^{-1}B' v_{ss} - x_r' Q x_r, \qquad w_{ss}(T) = 0 \quad (34)$$

Note that the final condition is given, then we have to solve this equation backwards in time. To avoid this we will derive an expression for the initial condition $w_{ss}(0)$. Using equations (4), (17), and (32) we have
$$v_{ss}' = z'\mathcal{T}_2'; \qquad x_r' = z' C_r' \mathcal{L}$$
substituting these equations in (34) we obtain
$$-\dot{w}_{ss} = z'\,[\mathcal{T}_2' BR^{-1}B'\,\mathcal{T}_2 - C_r'\mathcal{L}' Q_r]\,z$$
Because $z = e^{A_r t} z(0)$; $z' = z'(0)\, e^{A_r' t}$ we have
$$\dot{w}_{ss} = z'(0)\, e^{A_r' t}\, S\, e^{A_r t}\, z(0)$$
where
$$S = \mathcal{T}_2' BR^{-1}B'\,\mathcal{T}_2 - C_r'\mathcal{L}' Q_r \quad (35)$$
To find a solution for w_{ss} we assume a quadratic form
$$w_{ss} = z' S_{ss} z \quad (36)$$
then,
$$\dot{w}_{ss} = \dot{z}' S_{ss} z + z' S_{ss}\dot{z} + z'\dot{S}_{ss} z \quad (37)$$
considering S_{ss} a constant matrix, then $\dot{S}_{ss} = 0$.
Using $\dot{z} = A_r e^{A_r t} z(0)$; $\quad \dot{z}' = z'(0)\, e^{A_r' t} A_r'$ we obtain
$$\dot{w}_{ss} = z'(0)\, e^{A_r' t} A_r' S_{ss} e^{A_r t} z(0) + z'(0)\, e^{A_r' t} S_{ss} A_r e^{A_r t} z(0)$$

$$\dot{w}_{ss} = z'(0)\, e^{A_r' t}[A_r' S_{ss} + S_{ss} A_r] e^{A_r t} z(0) \quad (38)$$
Our assumtion of w_{ss} being a quadratic form will be correct if S_{ss} satisfies the Lyapunov equation
$$S = A_r' S_{ss} + S_{ss} A_r \quad (39)$$

The initial condition $w_{ss}(0)$ can be evaluated using the equation
$$w_{ss}(0) = z'(0)\, S_{ss}\, z(0) \quad (40)$$
where S_{ss} satisfies the Lyapunov equation (39) and S is given by (35).

5. Application of the Infinite Horizon LQT to Vibration Quenching

In this section the method described in the previos sections is applied to the design and simulation of a real-time optimal controller to suppress vibrations in flexible beams.

5.1. A Flexible Beam

Consider a flexible link constrained to move in its horizontal plane (no gravity effects) and actuated by a torque input (DC motor) at the hub. Using an overdot to denote time differentiation, the Euler-Bernoulli equations of motion and associated boundary conditions are given by
$$EI\frac{\partial^4 y(x,t)}{\partial x^4} + \rho\frac{\partial^2 y(x,t)}{\partial t^2} + \rho x\ddot{\theta}(t) = 0 \quad (41)$$
$$(J_b + J_h)\ddot{\theta}(t) + \rho\int_0^L x\frac{\partial^2 y(x,t)}{\partial t^2}dx = \tau(t) \quad (42)$$
$$y(0,t) = \frac{\partial y}{\partial x}(0,t) = \frac{\partial^2 y}{\partial x^2}(L,t) = \frac{\partial^3 y}{\partial x^3}(L,t) = 0$$
where $y(x,t)$ is the link's transverse displacement with reference to a rotating frame fixed to the hub; $\theta(t)$ is the angle between the hub's frame and a global (stationary) reference frame; EI is the bending stiffness; ρ is the linear mass density of the material; J_b is the flexible link's mass moment of inertia; J_h is the rigid or hub mass moment of inertia; L is the undeformed link's length; and $\tau(t)$ is the input torque applied at the hub. We assume that the link is homogeneous, has constant cross sectional area and that we allow only small displacements to remain within the linear regime.

The method of separation of variables leads to an infinite mode-expansion that, for practical reasons, is truncated to the first $N+1$ terms
$$y(x,t) = \sum_{j=0}^{N}\psi_j(x) q_j(t)$$
$$\theta(t) = \sum_{j=0}^{N}\theta_j q_j(t) = \sum_{j=0}^{N}\frac{d\psi_j}{dx}(0) q_j(t),$$
where $\psi_j(x)$ and θ_j are the j^{th} unconstrained mode shapes and $q_j(t)$ is the associated mode [5]. Substituting the truncated solution into equations (41), (42) and using an appropriately defined orthogonal relation among the mode shapes, yields the set of $N+1$ linear and uncoupled second order differential equations
$$M\begin{bmatrix}\ddot{\theta}\\ \ddot{q}\end{bmatrix} + \mathcal{K}\begin{bmatrix}\theta\\ q\end{bmatrix} = b\,\tau \quad (43)$$

where $M = M' > 0$, is the mass or inertia matrix, $\mathcal{K} = \mathcal{K}' \geq 0$ is the stiffness matrix, $b = [1\ 0 \cdots 0]'$, and q is the N-dimensional vector of flexible modes. Defining the (2N+2)-dimensional state vector

$$x = [\theta \quad \dot{\theta} \quad q_1 \quad \dot{q}_1 \quad \cdots q_N \quad \dot{q}_N]' \tag{44}$$

we assemble the state space equation

$$\dot{x} = Ax + Bu \tag{45}$$

$A \in \mathbb{R}^{n\times n}$, $x \in \mathbb{R}^n$, $B \in \mathbb{R}^{n\times 1}$, $u = \tau \in \mathbb{R}$ and $n = (2N+2)$.

5.2. Simulation Example

Using the beam's properties given in table 1, and considering a two modes model we obtain the following matrices:

$$\begin{aligned} M &= \begin{bmatrix} 1.3925 & 1.1028 & 0.1760 \\ 1.1028 & 0.9694 & 0 \\ 0.1760 & 0 & 0.9694 \end{bmatrix}; \\ \mathcal{K} &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3.5519 & 0 \\ 0 & 0 & 139.497 \end{bmatrix} \end{aligned} \tag{46}$$

After defining the state vector as in (44), we obtain

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 38.1425 & 0 & 239.0350 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -47.0569 & 0 & -271.9385 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -6.9241 & 0 & -187.2933 & 0 \end{bmatrix};$$

$$B = [0 \quad 9.4393 \quad 0 \quad -10.7386 \quad 0 \quad -1.7135]'$$

The simulation of the open loop system is given in fig. 1. Choosing the weight matrices, $Q_1 = 0$, $Q_2 = 100$ in eq. (4) we obtain

$Q = diag(100 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0)$,

$\mathcal{L} = [1 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0]'$

and the closed-loop gain

$A_c = [A - BK_{ss}] =$

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -94.39 & -33.77 & 15.41 & -18.23 & 203.68 & -10.43 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 107.39 & 38.42 & -21.20 & 20.74 & -231.72 & 11.86 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 17.14 & 6.13 & -2.80 & 3.31 & -180.88 & 1.90 \end{bmatrix}$$

$$K_{ss} = [10 \quad 3.5782 \quad 2.4079 \quad 1.9310 \quad 3.7453 \quad 1.1049]$$

We want to track an exponentially decaying signal

$$y_r = Csin(\omega_n t)e^{-w_n t}$$

Then our reference system can be represented by

$$\begin{aligned} \dot{z} &= \begin{bmatrix} 0 & 1 \\ -\omega_n^2 & -2\zeta\omega_n \end{bmatrix} z \\ y_r &= C_r z \end{aligned} \tag{47}$$

We compute $Q_r = Q\mathcal{L}C_r$

$$Q_r = \begin{bmatrix} 100 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}'$$

Then we solve the matrix equation (27). Note that
$\sigma(A_c) = [1.043 \pm 13.735i; 4.123 \pm 7.794i; 0.406 \pm 1.791i]$;
$\sigma(A_r) = [-0.1 \pm 0.995i]$; and this satisfies
$\sigma(A_r) \cap \sigma(A_c) = 0$
The solution of the matrix equation is:

$$\mathcal{T}_2 = \begin{bmatrix} 41.1693 & 12.4286 \\ 3.3800 & 9.8348 \\ 25.0912 & 7.1565 \\ 1.9495 & 8.1813 \\ 10.3228 & 3.1119 \\ 1.2915 & 0.6475 \end{bmatrix}$$

To simulate the infinite horizon LQT we have the following equations

$$\begin{aligned} u(t) &= -K_{ss}x(t) + R^{-1}B'v_{ss} \\ v_{ss} &= \mathcal{T}_2 z \end{aligned} \tag{48}$$

using the initial values:
$x(0) = [22 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0]'$, $z(0) = [5.0 \quad -0.5]'$

Using Simulink for the simulation, we obtained the results given in figures 2, 3 and 4. In fig. 2 the reference and the output signals are given. Fig. 3 shows the control input that was applied to the flexible structure in order to track the desired trajectory $y_r(t)$. Fig. 4 shows the functional cost. For this example we obtained the minimum cost $J^* = 2.2E4$. See [6] for details.

6. Conclusions

This article presented the infinite horizon linear quadratic tracker problem from the model-following viewpoint. A design methodology is proposed for asymptotically stable reference signals, and it is shown that for these cases, the cost is bounded and an optimal controller can be obtained. The results show the effectiveness of the method.

Table 1. Flexible Beam Parameters

Linear Mass Density	$\rho = 0.4847\,kg\,m^{-3}$
Length	$L = 2.0\,m$
Flexible Beam Inertia	$J_b = \rho L^3/3$
Hub Inertia	$J_h = 0.1\,kg\,m^2$
Area Moment of Inertia	$I = 3.3339e-11\,kg\,m^2$
Young's Modulus	$E = 6.8944e+10\,N\,m^{-2}$

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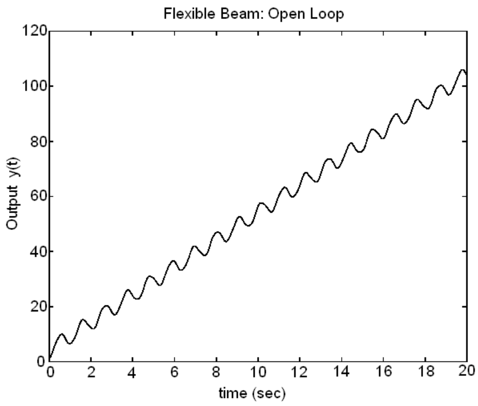

Figure 1. Open-loop System

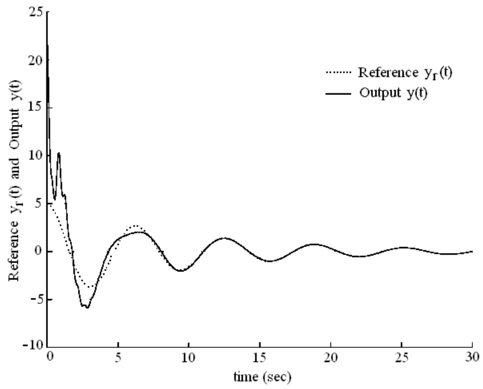

Figure 2. Reference and Output Signals

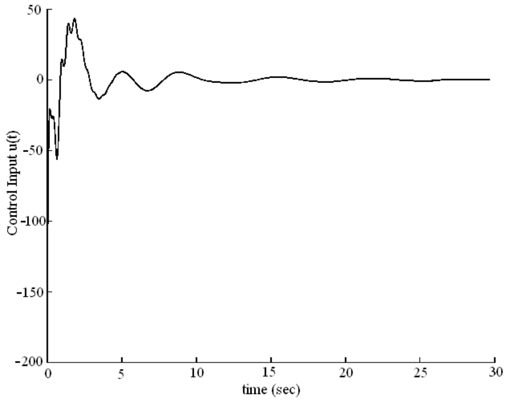

Figure 3. Control Input u(t) (two-mode model)

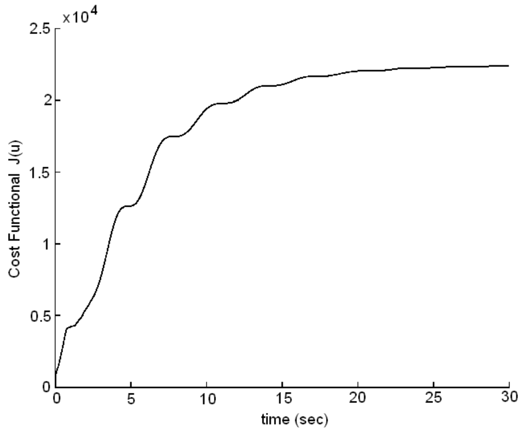

Figure 4. Cost Functional