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Quasioptimal Sliding Mode Controller for a Flexible Structure

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Abstract— **In this paper, we describe the combined design of a time-optimal and a sliding mode controller for a single-axis, N-mode model of a flexible slewing structure, where the main goal is the implementation of a fast controller. A model consisting of an aluminum flexible link with torque actuation provided by a DC motor at its hub is constructed in modal coordinates and a recently introduced time-optimal torque law for a one-mode model of this system that can be implemented in real time is used. The sliding-mode controller is designed so that the torque generated by a DC motor converges to the desired time-optimal torque law. After the time-optimal control law has been tracked within a neighborhood of the origin, a full-state feedback controller is used to drive the system's elastic modes exactly to the origin in finite time. The optimal gain vector for this terminal controller is the solution to an LQR problem. An observer is designed to estimate the unknown system states from actual measurements of tip-position, angle, and speed of the flexible structure, where its gain is obtained by the pole placement technique. Computer simulations of the proposed controller are shown in this work to illustrate the design idea. In the simulations, a one mode model system of the flexible slewing structure is considered first, then the controller is applied to a model of the system that includes higher-order flexible modes.**

I. INTRODUCTION

Research in modeling and control of flexible structures has always been of interest, particularly in maneuvering space structures and large robots. The subject of flexible structure control with a maneuver-time minimization constraint is of main concern. In order to decrease the complexity of the problem, sequential single axis maneuvers are usually preferred over simultaneous three-axis slews.

A typical single axis linear model of a flexible structure consists of a *rigid mode* describing the motion of the body as if it were rigid, and N *flexible modes* describing the vibrational motion caused by the distributed elasticity. Motivation in using flexible manipulators arises from their ability for high speed maneuvers, less weight, higher mobility, reduced energy consumption, and lower inertial forces for accurate positioning. However, a serious disadvantage of using light, flexible arms is that any vibrational transients in the flexible arm during rapid movements should be eliminated.

In this paper, the main goal is to control the rest-to-rest movement of a flexible link from an initial position to a final position in minimum-time. We analyze a representative flexible structure comprised of a flexible link that uses a DC motor as the source of actuation, where its model is expressed as a set of partial differential equations dynamically coupled to the motor by a set of linear ordinary differential equations. The control design methodology is divided in two steps: (1) assuming that the control input is the torque, a nominal or desired control law is designed using a recently introduced time-optimal control scheme; (2) the actuator dynamics are introduced and a sliding mode controller is designed so that the generated torque matches the desired or nominal control signal. The design framework exploits the block structure of the system equations.

Different aspects of the time-optimal control problem for flexible structures are discussed in [1]-[7]. *Continuous* control laws that quickly maneuver flexible structures through a large angle while minimizing some performance criterion have been obtained. *Discontinuous* actuation (but not time-optimal) has been investigated in the control of flexible satellites. Some experimental structures have been used to test rigid-body time-optimal controllers, that is, it is assumed that no vibrational modes are present. However, such a controller can destabilize the flexible modes and lead to structural failure. Another technique that has been employed to generate a rapid slew maneuver is an optimal, open-loop, rigid-body control with a perturbation feedback controller for vibration suppression. We emphasize that all of these methods result at best in a *near minimum-time* behavior.

Typically, the actuator dynamics are not included in the model since it considerably complicates the derivation of the optimal switching strategy. Therefore, it is usually assumed that the actuator dynamics are very fast and can be neglected. The end result is that the control law is derived for an actuator that directly produces a torque input. In this paper, we illustrate how the actuator dynamics can be re-introduced after the time-optimal scheme has been derived, and a fast sliding mode controller is designed so that the actuator's output closely follows the desired time-optimal control law.

The minimum-time problem for the rigid mode and N flexible modes has been the focus of several articles. The article [6, 7] uses a one-mode model of a flexible beam to determine the switching times by a phase-plane analysis technique. This technique is based on solving three equations on-line to get the optimal switching strategy. Its structural frequencies, elements of the block diagonal system matrix A, depend on several properties of the beam such as, length, linear density of the material, and beam moment of rotational inertia. In [6], the first known description of the switching hypersurface for a model of a flexible structure was introduced. The control law was developed via phase-plane analysis, thus constructing the optimal switching strategy in

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terms of the system states. We will refer to it as the *nominal* or *desired* control strategy.

II. Model and Nominal Control of a Flexible Link

In [1], the equations of motion for a flexible link with torque actuation at its hub were derived. Figure 1 illustrates the transverse displacement in the movement of a flexible beam denoted by $y(x,t)$ and measured with respect to a rotating reference frame, $\theta(t)$ the angle of rotation of the hub, and τ_ι the torque delivered to the link.

An N-mode expansion for the displacement described as

$$y(x,t) = \sum_{i=1}^{N} \psi_i(x) r_i(t) \ , \quad x \in [0, L] \ , \quad t \geq 0 \qquad (1)$$

where the spatial-dependent functions $\psi_i(x)$ are the mode shapes, and the time-dependent functions $r_i(t)$ are the modes, is substituted into the equations of motion, and leads to the finite-dimensional model

$$\begin{bmatrix} \ddot{\theta} \\ \ddot{r} \end{bmatrix} + M^{-1} K \begin{bmatrix} \theta \\ r \end{bmatrix} = M^{-1} b_\iota \tau \qquad (2)$$

where M is the symmetric, positive-definite mass matrix, K is the symmetric, positive semidefinite stiffness matrix, and $b_\iota = [1 \quad 0 \quad \cdots \quad 0]^T$.

Taking into account the inertia of the electric motor hub

$$\tau_\iota = \tau - J_m \ddot{\theta} \quad ; \quad \tau = K_\tau I_m \qquad (3)$$

where $\ddot{\theta}$ is acceleration, τ the torque provided by the motor, and τ_ι the load torque, the modal state vector is defined as

$$x \ = \ [x_0^1 \ \ x_0^2 \ \ x_1^1 \ \ x_1^2 \ \ x_2^1 \ \ x_2^2 \ \ \cdots \ \ x_n^1 \ \ x_n^2]^T.$$

In this way, we assemble the state space equations as

$$\dot{x} = Ax \ + \ b\tau \qquad (4)$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$ and matrices A and b have a form

$$A \ = \ \text{blockdiag} \begin{bmatrix} 0 & 1 \\ -\omega_i^2 & 0 \end{bmatrix}, \quad i = 0, 1, 2, \ldots, n \ ;$$

$$b \ = \ [0 \ \ b_0 \ \ 0 \ \ b_1 \ \ 0 \ \ b_2 \ \ \cdots \ \ 0 \ \ b_n]^T$$

which is suitable for a variety of controller design techniques.

Specifically, we consider a one-mode model without damping given by

$$\dot{x} = Ax \ + \ b\tau \ , \qquad x \ = \ [x_0^1 \ \ x_0^2 \ \ x_1^1 \ \ x_1^2]^T \qquad (5)$$

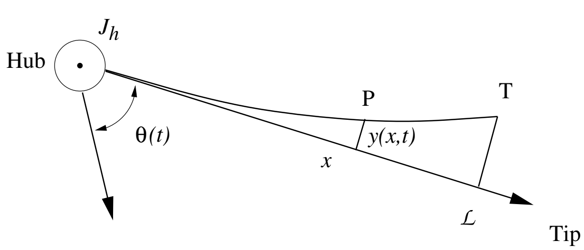

Fig. 1. Deflection in a flexible beam

where

$$A \ = \ \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\omega_0^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -\omega_1^2 & 0 \end{bmatrix}, \quad b \ = \ [0 \ \ b_0 \ \ 0 \ \ b_1]^T.$$

$0 = \omega_0 < \omega_1$ are the two structural frequencies representing the *rigid* and *flexible* modes, respectively, and $x \in \mathbb{R}^4$ is the state consisting of the *rigid mode* angle and speed, and the *flexible mode* position and speed.

The *rest-to-rest minimum-time slewing problem* is to find the control law constrained by $|\tau(t)| \leq 1$ that will steer (5) from the initial rest position $x(0) = [\theta_0 \quad 0 \quad 0 \quad 0]^T$ to the final rest position $x(t_f) = [0 \quad 0 \quad 0 \quad 0]^T$ such that the cost $J = t_f$ is minimized.

The optimal bang-bang control $\tau^*(t)$ which solves this problem has been shown to exhibit the following three main characteristics: [6]

- $\tau^*(t)$ is an odd function of time with respect to the mid-maneuver time $t = t_f^*/2$.
- The rigid-mode phase-plane trajectory in the (x_0^1, x_0^2)-plane is symmetric with respect to the line $x_0^1 = \theta_0/2$; and the flexible mode phase-plane trajectory in the $(\omega_1 x_1^1, x_1^2)$-plane is symmetric with respect to the line $\omega_1 x_1^1 = 0$.
- For the case of one flexible mode, that is, system (5), the optimal control switches between -1 and $+1$ *at most three times*.

A characterization of the switching law $\tau^*(t) = \{\tau_4, \ \tau_3, \ \tau_2, \ \tau_1\}$ was developed using phase-plane analysis techniques. The result was the first known description of the switching hypersurface that (i) does not require the off-line computation of the switching time instants, (ii) it can be implemented in real time, and (iii) it requires only the rigid-body states. The switching strategy is uniquely characterized by the following three equations: [7]

$$\cos(\sigma_2 + 2\sigma_3) - 2\cos(\sigma_2 + \sigma_3) + 1 = 0 \qquad (6)$$

$$-\sin(\sigma_2 + 2\sigma_3) + 2\sin(\sigma_2 + \sigma_3) + 2\sqrt{1 - \cos(\sigma_3)} = 0 \quad (7)$$

$$\sigma_2 + \sigma_3 \geq -\frac{\pi}{2} \qquad (8)$$

where

$$\sigma_2 \triangleq \frac{\tau_2 \omega_1}{b_0} \, x_{0,2}^2 \qquad ; \qquad \sigma_3 \triangleq \frac{\tau_3 \omega_1}{b_0} \, x_{0,3.}^2$$

The variables σ_2 and σ_3 depend only on the rigid body speed state variable. An algorithm is described in [6] which allows the real-time implementation of the optimal control $\tau^*(t)$.

III. Controller and Observer Design

In this section, we describe the controller design that directs the states of an N-mode model of a flexible structure from an initial position to the origin in near-minimum time. The proposed controller consists of two stages: (1) a sliding mode controller that closely tracks the nominal control law given by a time-optimal controller for a one mode model of a flexible structure which reaches an ε neighborhood of the origin, and (2) a full-state feedback terminal controller based on solving the quadratic regulator problem (LQR), that guides the system states asymptotically to the origin. In addition, accounting for unavailable states, we also design an observer that provides their estimated value.

A. Sliding mode control design

Considering that sliding mode control presents a good strategy in solving the problem of tracking and modeling of imprecise systems, a controller is designed to accurately track the states of the nominal control scheme for a one bending mode model of a flexible slewing structure. The design is first made assuming that the *rigid body states* are available for controller design. Therefore, as the nominal switching strategy is dependent on the rigid body states only, these modal states will be estimated by an observer based on the output's available measurements of motor angle, speed, and link-tip position.

The reason we are advocating sliding mode control is that it is relatively easy to implement, it has proven tolerances to external disturbances and model uncertainties, and very importantly, it efficiently solves the problem of how to re-introduce the actuator dynamics after a particular control design has been completed without the need for redesign while maintaining a similar and quite acceptable level of performance.

The current equation (9), describing the dynamics of the motor is

$$L\dot{I}_m + RI_m + K_b\dot{\theta} = V_a = u \tag{9}$$

where L, R, and u are the armature inductance, resistance and input voltage, respectively. Assuming that the first mechanical equation and the current equation of the DC motor shown in equations (3) and (9) describe its model, and considering that (9) represents the actuator dynamics, in addition to the fact that the control input is defined as the torque provided by the motor as is shown in (3), we propose to design a sliding mode controller based on the following definition of the sliding manifold

$$S(t) \stackrel{\triangle}{=} \tau \; - \; \tau^* \tag{10}$$

Equation (9) is used to design the input voltage as,

$$u = u_{eq} \; + \; u_d \tag{11}$$

where u_{eq} is the *equivalent control* obtained by forcing $\dot{S} = 0$, and u_d is the discontinuous control that maintains the system states in the desired sliding manifold.

The equivalent control u_{eq} is not used here because the theoretical $\dot{I}_m^*$ involves delta functions at the switching instants. Therefore, the control input to the system is only given as the discontinuous control

$$u \;=\; -M_0 \text{ sign } (S) \tag{12}$$

where the gain M_0 is chosen large enough to ensure that sliding occurs.

In the design of the gain M_0, we must consider that the nominal time-optimal control τ^* presents discontinuities at the switching instants, therefore, in the following analysis the transitions from u^- to u^+ (switching) is assumed to have a finite slope. This is represented by a linear transition in each of the switchings instead of an instantaneous motion as is done theoretically. A sketch showing the different switchings is seen in Figure 2.

Taking the derivative of the sliding surface S and substituting $\dot{I}_m$ and then V_a from the current equation and the sliding control respectively leads to the following developments. The derivative of the sliding surface equation is

$$\dot{S}(t) = K_\tau(\dot{I}_m - \dot{I}_m^*) \tag{13}$$

As I_m^* remains constant during the time interval, its derivative $\dot{I}_m^*$ is zero for that interval. Let such an interval of time be denoted $S_j(t_{j-1}, t_j)$. Equation (13) is simplified to

$$t_{j-1} \; < \; t \; < t_j \;\; : \;\; \dot{S}(t) = K_\tau \dot{I}_m \tag{14}$$

$$\dot{S}(t) = \frac{K_\tau}{L}\left(-M_0 \text{ sign } (S) \; - \; RI_m \; - \; K_b\dot{\theta}\right) \tag{15}$$

$$\dot{S}(t) = -\frac{K_\tau}{L} M_0 \text{ sign } (S) + S_1 \tag{16}$$

In the time interval

$$t_j + \epsilon \; < \; t \; < \; t_j + \epsilon \tag{17}$$

$$\dot{S}(t) = -\frac{K_\tau}{L} M_0 \text{ sign } (S) - m_j \tag{18}$$

where the switching slope is m_j.

Theoretically M_{0j} should be selected individually for each time interval. This gain should be large enough to make S_1 negligible and for the signs in $\dot{S}$ and S to be opposite (because of the $-M_0$ gain), to direct the state trajectories to the surface S. The gain M_0 is set to the largest M_{0j}. However, in the simulations only one gain was chosen. This gain corresponds to the voltage being delivered to the motor.

The sliding mode control system that has been designed only drives the system states within a neighborhood of the origin, therefore, a terminal controller that directs the states asymptotically to the origin is also needed. This terminal control which is given by a full-state feedback linear controller will be described in the following paragraphs.

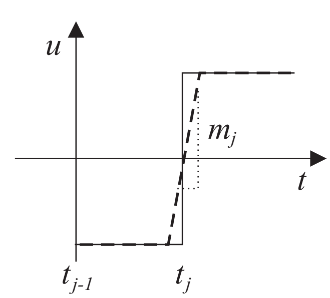

Fig. 2. Instantaneous switching (solid line) vs. linear transition switching (dotted line)

B. Terminal controller design

Once the flexible beam has reached the origin, it will oscillate depending on the number of modes that have been considered. In order to stabilize the tip-position oscillations in the flexible beam, a terminal control is designed as a full-state feedback control law where the gain is obtained as the solution to the regulator problem. The control is switched when the angle of maneuver reaches an ε neighborhood of the origin. This feedback control will suppress the movement in the structure which asymptotically decays to rest.

Under the condition is that $\{A, b\}$ is controllable and considering the cost is $J = \int_{t_0}^{t_f} [x^T(t)Qx(t) + ru^2(t)]dt$, where Q and r determine the penalization, the feedback control is given by a full state feedback law $u(t) = -kx(t)$, in which optimal gain k is found by solving the Riccati equation in the regulator problem for a given set of matrices $\{A, b, Q, r\}$. In this way, the feedback controller gains are designed to quickly direct the states to the origin. The asymptotic stabilization of the closed-loop system is guaranteed by the optimal feedback design.

C. Observer design

Control design is usually made assuming that all the states are available for measurement. In this paper, the controller design is initially made for the case assuming that all the system states are known. However, in practice the required state is not available for measurement, an observer is designed to estimate the required unknown state vector needed in the nominal control scheme from available measurements of angle, velocity and tip-position of the flexible beam.

We verify the observability matrix to be $\mathrm{rank}(A, C) = n$, and select the poles of the system to certain desired values by using the *pole placement* method. Therefore, we design the observer as the set of state space equations described by

$$\dot{\hat{x}} = (A - LC)\hat{x} \ + \ [b \quad L] \begin{bmatrix} u \\ y \end{bmatrix} \tag{19}$$

where $b \in \mathbb{R}^{n \times m}$, and $L \in \mathbb{R}^{n \times r}$. The correct selection of the gain L is crucial for the estimated states to converge to the measured ones accurately enough before the nominal minimum-time controller switchings are performed.

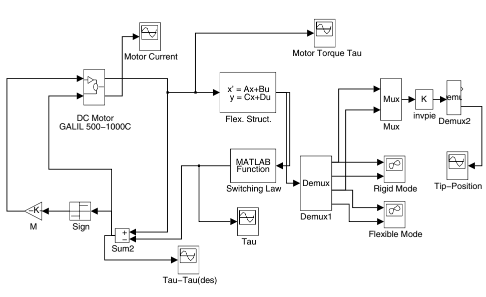

Fig. 3. SIMULINK diagram of flexible structure with sliding mode controller where the switching equations are (6), (7), and (8)

IV. SIMULATIONS

In this section, the behavior of a controller is analyzed by performing a series of simulations of the dynamic system.

A. Non-observer based controller system

Figure 3 shows a SIMULINK diagram of the overall system including the motor dynamics, the nominal torque control generator, and the sliding mode controller. It describes a non-observer system which is built in the assumption that the system states are known. In the simulations, the numerical values used for parameters R, V, and I_m have been chosen as listed in the specification sheets of the GALIL 500-1000C DC motor, however, a much larger value of motor inductance was used in order to avoid simulation step-size problems.

A one-mode model of the flexible link was assembled using constrained-mode expansions. The time-optimal algorithm was implemented in order to generate the *nominal* torque signal. Figure 4 illustrates the resulting sub-optimal torque history, and tip-position. The plots show how the system resembles a fast sliding mode controller which results in a quasi time-optimal control. Observe that the motor is indeed controlled very fast to track the desired time-optimal torque signal. Both the motor torque and current are within the specification sheet bounds.

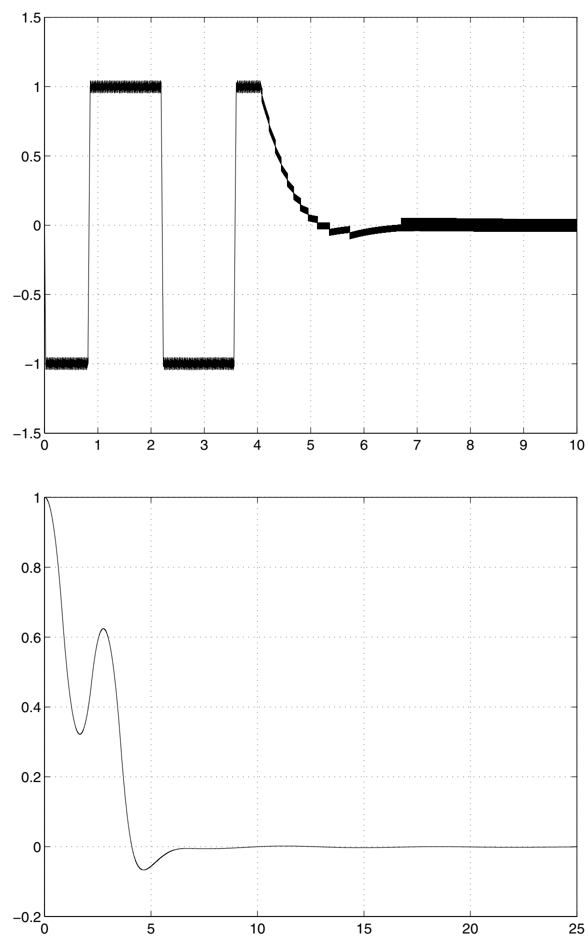

Fig. 4. Sliding control torque and (Tip-deflection)tip-position plots for one mode model

B. Observer based controller system

Figure 5 shows the SIMULINK diagram describing the complete observer based controller system for the rest-to-rest movement based upon online estimates of the flexible structure modes from available measurements of velocity, tip position and angle of the motor. In the plant, the output system matrix C is expressed for the estimated states as

$$y(x,t) = \sum_{i=1}^{N} \psi_i(x) r_i(t) \tag{20}$$

MATLAB is used to calculate this function by using a program which generates the mode shapes ψ_i for an N-mode flexible beam. The tip-position of the beam is obtained from (20) at $x = \mathcal{L}$, where $\mathcal{L}$ is the length of the beam.

$$y(\mathcal{L},t) = \sum_{i=1}^{N} \psi_i(\mathcal{L}) r_i(t) \tag{21}$$

The state of an N-mode flexible beam is usually defined as

$$x = \begin{bmatrix} \theta & \dot{\theta} & r_1 & \dot{r}_1 & r_2 & \dot{r}_2 & \cdots & r_N & \dot{r}_N \end{bmatrix}^T \tag{22}$$

Therefore, according to this order, the vector y is expressed

$$y(\mathcal{L},t) = [0 \quad 0 \quad \psi_1(\mathcal{L}) \quad 0 \quad \psi_2(\mathcal{L}) \quad 0 \quad \cdots \quad \psi_N(\mathcal{L})]. \tag{23}$$

Simulations are shown in Figure 8.

C. Application to a higher-dimensional model

The proposed controller designed for a one mode model of a flexible structure is used in models that consider more flexible modes. Figure 6 illustrates a flexible beam divided into 4 elements and 4 nodes (y_1, y_2, y_3, y_4).

The model of the flexible structure must include damping, which should be considered before transforming the system to modal coordinates. The result is a *damped* model of the flexible structure with motor current as the control input. As the minimum-time control law for the *damped* model has not yet been developed, the application of τ^* will inevitably result, at best, in time suboptimal operation. However, since the natural damping is small, we expect the near minimum-time behavior to be satisfactory.

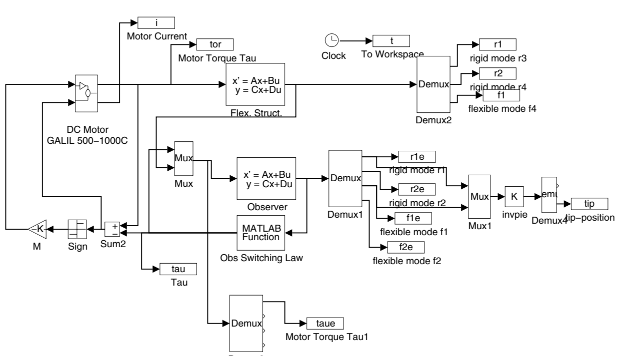

Fig. 5. SIMULINK diagram of flexible structure with observer based sliding mode controller where the switching equations are (6), (7), and (8)

V. CONCLUSIONS

In this paper we analyzed the problem of introducing the actuator dynamics *after* a nominal control design has been completed. The motivation is that, in certain situations, the motor dynamics complicate the nominal control design considerably. Such is the case in the design of time-optimal controllers for flexible structures. We have successfully simulated a time-suboptimal control law for an N-mode model of a flexible link with torque actuation at its hub. The DC motor is controlled by a sliding mode controller that follows a *nominal* torque signal and which effectively forces the motor to closely track the desired time-optimal torque law. This provides that the motor is indeed controlled fast enough to track the desired time-optimal torque signal. Simulations have shown that this is indeed a viable solution that reduces the cost associated to controlling the higher dimensional model in that way simplifying the control.

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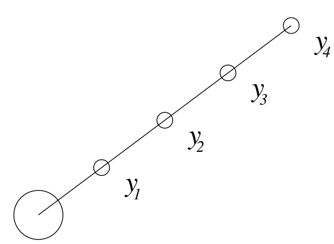

Fig. 6. Representation of the flexible structure in nodal coordinates

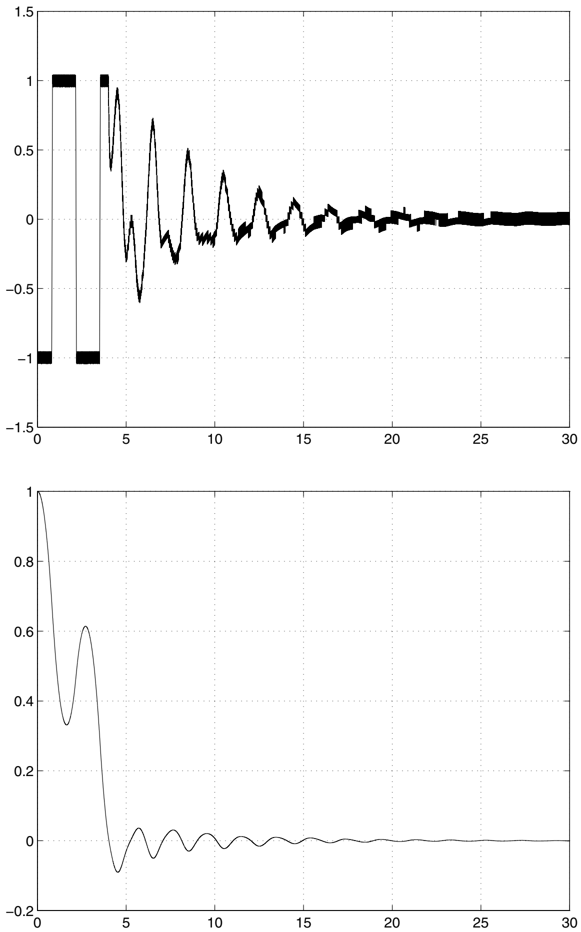

Fig. 7. Sliding control torque and tip-position angle plots for a four mode model flexible structure

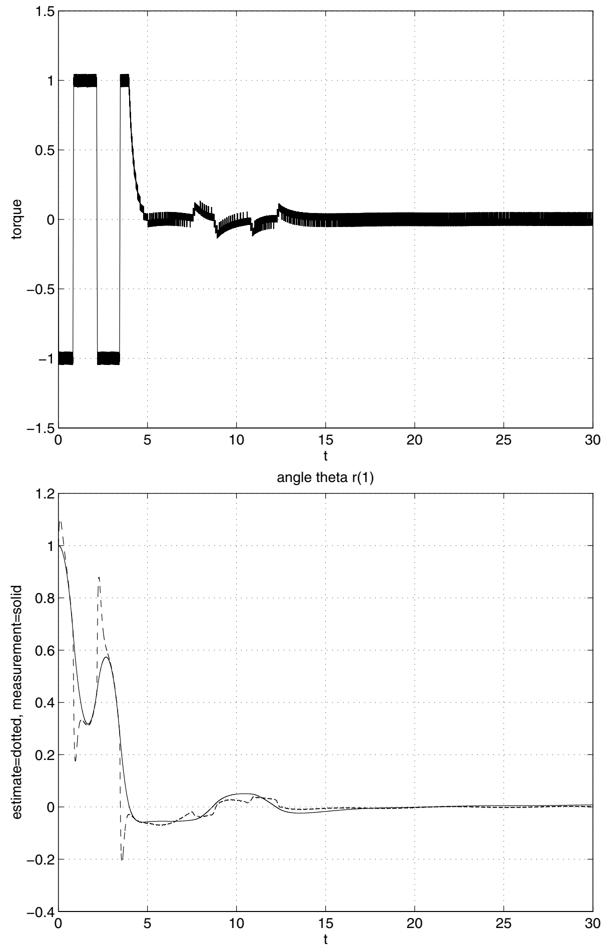

Fig. 8. Observed sliding torque control signal and observed tip-position (solid=measurement, dotted line=estimated value)